The Ladder of Time D: Scale Factor in the Time Hierarchy – Algebraic Rigidity, Causal Cycle Integrality, and Cosmological Observational Tests

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Abstract

In standard inflationary theory, the total number of e-folds of accelerated expansion in the early universe is usually freely adjusted by the shape of the inflaton potential, lacking a rigid lock at the fundamental level. Starting from the hierarchical theory of time structure, this paper demonstrates that there exists a dimensionless scale factor in the transitions between time levels, whose numerical value is uniquely determined as $r = 2$ by the dimension-doubling law of the Cayley-Dickson construction for Hurwitz algebras, without introducing any free parameters. By establishing a "causal-metric isomorphism" correspondence principle, the algebraic doubling law of information capacity is mapped to the discrete growth of the inflationary scale factor, leading to the strict scaling relation $N_e = n_c \ln 2$ between the total number of e-folds N_e and the number of causal cycle closures n_c. The atomicity and integrality of causal cycle closures require n_c to be a positive integer, while the number of macroscopic level transitions $k = 3$ in the time hierarchy further demands that n_c be an integer multiple of 3. By combining this double arithmetic condition with the phenomenological constraints of standard cosmology on the horizon problem, the allowed values of n_c are compressed to three discrete candidates, yielding a rigid prediction interval $N_e = 54.1 \pm 2.0$ centered on $n_c = 78$. This paper further reveals that the inherent incomplete closure effect during the critical stage when the time hierarchy approaches the classical time level necessarily leaves a characteristic quasi-sinusoidal modulation imprint in the temperature angular power spectrum of the cosmic microwave background. The multipole position of this modulation is uniquely determined as $l_* \approx 26 \pm 2$ by the number of microscopic closures per macroscopic transition $n_s = 26$, with a relative amplitude of about 1–2%. A statistical comparison with Planck 2018 data shows that the predicted interval is highly consistent with observational constraints, and the modulation position overlaps with known low-l anomaly regions. This modulation prediction, together with the scaling constraint on N_e, constitutes a dual-channel independent test, providing a clear falsifiable path for the time hierarchy theory.

Keywords: Time hierarchy; Scale factor; Causal cycle; Inflationary e-folds; CMB power spectrum; Hurwitz algebras; Cayley-Dickson construction; Causal-metric isomorphism

1 Introduction

Cosmology has long faced a fundamental question: is the total number of e-folds N_e determining the duration of accelerated expansion in the early universe fundamental or derivative? In the standard single-field slow-roll inflation framework, the range $N_e \approx 50$–60 is determined by the slope of the inflaton potential and coupling constants, and the theory itself does not provide a rigid lock on the specific numerical value. The pioneering work of Guth (1981) and the chaotic inflation model of Linde (1983) established the inflationary paradigm, but the precise value of the number of e-folds still depends on the parametric choice of the potential shape. This freedom of adjustment makes inflation theory highly adaptable to observations but also weakens its predictive power – any value falling within the 50–60 range can be explained a posteriori.

Meanwhile, gravitational thermodynamics reveals another scaling phenomenon. Gibbons and Hawking (1977) proved that the cosmological horizon of de Sitter spacetime has a temperature proportional to the Hubble parameter, suggesting that the causal horizon is not a purely geometric boundary but a physical entity with information encoding capacity. The horizon entropy is given by the Bekenstein-Hawking formula, and its value is proportional to the horizon area. Jacobson (1995) further interpreted the Einstein equation as a thermodynamic equation of state, revealing a dynamical link between spacetime expansion and entropy change. This leads to a deeper question: is the growth of horizon entropy driven by some discrete, non-adjustable information doubling mechanism?

Could it be that the above two seemingly independent problems – the non-rigid origin of the inflationary e-folds and the possibility of discrete growth of horizon entropy – share a single underlying scaling structure? This question constitutes the starting point of the present investigation.

Starting from the hierarchical theory of time structure, this paper argues that the physical essence of inflation corresponds to the hierarchical transition from a quantum disordered state to a classical causal order, and that the exponential growth of the scale factor is driven by the discrete closure of causal cycles. There exists a categorical equivalence preserved by a faithful functor between the sequence of time levels and the sequence of Hurwitz algebras; the dimension-doubling law of the latter under the Cayley-Dickson construction is uniquely determined by the theorem of Hurwitz (1898) as a strict doubling by a factor of 2, thereby locking the scale factor $r = 2$ as an inevitable consequence of algebraic rigidity. This scale factor contains no adjustable parameters and is the cornerstone of the entire theory. On this basis, the discreteness and integrality of causal cycle closures constrain the observable inflationary e-folds to a very narrow interval and further predict a characteristic modulation pattern in the CMB power spectrum. The entire logical chain of this paper proceeds from the algebraic structure, through the doubling of information capacity and the integrality of causal cycles, and finally reaches cosmological predictions that can be directly tested by CMB observations, forming an unbroken reasoning path from underlying mathematical rigidity to top-level physical observations.

2 Information Capacity of Time Levels and the Inflationary Mapping

2.1 Operational Definition of Causal Order Types and Information Capacity

In the theory of time levels, each level corresponds to a set of causal order types that can be unambiguously distinguished within that level. Let the set of equivalence classes of these types be denoted by C_n, and let its cardinality be defined as the information capacity of that level, $I_n = |C_n|$. The operational criterion for this definition is as follows: two causal order types are distinguishable if and only if there exists a physical process that assigns them to different causal classifications in the description of the adjacent higher level $\mathcal{T}_{n+1}$. Indistinguishable causal order types are regarded as the same equivalence class within a given level. This criterion grounds the information capacity as a physical quantity testable by the causal logic of a higher level, thereby freeing it from any intuitive dependence on the term "information."

In the quantum disordered ground state, no distinguishable causal order structure exists, so $I_0 = 0$. As the time levels unfold step by step, the number of equivalence classes of causal order types strictly increases, and this increase is driven by the closure operation of causal cycles.

2.2 Algebraic Rigidity Locks the Doubling of Information Capacity

Between the sequence of time levels (generating time, parametric time, reference time, classical time) and the sequence of Hurwitz algebras (real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, octonions $\mathbb{O}$) there exists a categorical equivalence preserved by a faithful functor. Under this equivalence, the equivalence classes of causal orders at each level correspond to a set of basis elements in the corresponding algebra, and the information capacity is proportional to the real dimension of the algebra. Since the dimension-doubling law of the Cayley-Dickson construction is uniquely determined by the theorem of Hurwitz (1898) as $\dim A_{n+1} = 2 \cdot \dim A_n$, and since beyond the octonions no algebra satisfying the normed multiplication law exists, the growth of information capacity between adjacent time levels is rigidly locked by algebraic rigidity as

$$I_{n+1} = 2\, I_n.$$

This doubling relation contains no adjustable parameters and is a direct consequence of information theory and algebraic structure (a detailed proof is given in Appendix A). This result means that each complete objectification operation—i.e., the closure of a causal cycle—necessarily doubles the number of distinguishable causal order types exactly, and the scale factor $r = 2$ thereby becomes the rigid skeleton of the entire theoretical framework.

2.3 Causal Cycle Closure as the Discrete Driving of Information Capacity

A causal cycle closure is defined as an atomic operation consisting of three indivisible steps: enumerating the equivalence classes of causal order types at the lower level; constructing the dependency mapping between these equivalence classes; encoding this mapping in the form of a meta-description at the higher level and verifying its self-consistency. Once all three steps are completed, the lower-level causal structure is seamlessly elevated to an explicit object at the higher level, and the information capacity doubles. Therefore, causal cycle closure is the sole indivisible discrete event responsible for the growth of information capacity; its number of occurrences n_c must be a strictly positive integer, and the total information capacity is multiplied by 2^{n_c}.

In the complete sequence of time levels from the quantum disordered ground state to classical time, there are $k = 3$ macroscopic level transitions (generating $\to$ parametric $\to$ reference $\to$ classical). Each macroscopic transition consists of n_s microscopic causal cycle closures, so $n_c = 3n_s$, and consequently n_c must be an integer multiple of 3. This integrality condition will directly affect the allowed values of the inflationary e-folds derived later.

2.4 The Causal-Metric Mapping Postulate

To map the abstract doubling of information capacity onto the observable scale-factor expansion in cosmological inflation, a clear physical bridge must be established. In this section we introduce the *Causal-Metric Mapping Postulate*—a physical hypothesis that connects the discrete information structure of the time hierarchy to the continuous geometric evolution of spacetime. We explicitly distinguish its logical status from the algebraic rigidity results of Chapter 3: the postulate is a *bridge assumption* whose validity must ultimately be decided by observational tests, not derived from the Cayley-Dickson construction alone.

2.4.1 Statement of the postulate

During the quasi-de Sitter phase of inflation, the homogeneous and isotropic background establishes a correspondence between the conformal structure of spacetime and the counting of causal order types. The postulate asserts that each closure of a causal cycle manifests at the gravitational level as an atomistic expansion step, multiplying the cosmic scale factor $a(t)$ by a fixed factor. Specifically, the horizon entropy $S = A/4G$ [3] increases by $\ln 2$ for each independent causal degree of freedom generated, and the corresponding exponential increment of the scale factor satisfies

$$\Delta N_e = \ln 2, \tag{1}$$

where $N_e \equiv \ln(a_{\text{end}}/a_{\text{beg}})$ is the number of e-folds.

In the time-level framework, the closure of a single causal cycle produces exactly one bit of definite capacity in information space, doubling the number of independent causal domains on the horizon. Hence, under the postulate, the contribution of each closure to the inflationary e-folds is

$$e^{\Delta N_e} = 2, \qquad \Delta N_e = \ln 2. \tag{2}$$

After n_c closures, the total growth of the scale factor satisfies

$$e^{N_e} = 2^{n_c}, \quad \text{i.e.,} \quad N_e = n_c \ln 2. \tag{3}$$

2.4.2 Logical status and testability

The relation $I_{n+1} = 2I_n$ is a theorem of the categorical equivalence (Chapter 2), derived from the Cayley-Dickson dimension-doubling law. The relation $N_e = n_c \ln 2$, however, is *not* a theorem; it is the Causal-Metric Mapping Postulate translated into an equation. Its physical content is the claim that *information capacity doubling and cosmological scale-factor growth are locked at the horizon level.* This locking is motivated by the thermodynamic interpretation of the Einstein equations [10] and the Bekenstein-Hawking entropy formula, but it cannot be derived from pure algebra.

The postulate is falsifiable: if future precision measurements of the primordial power spectrum, CMB polarization, or large-scale structure determine that the effective number of e-folds is incompatible with any integer n_c satisfying the integrality conditions of Chapter 4, then the postulate would be refuted. Conversely, if the discrete predictions derived from the postulate are confirmed, the postulate gains empirical support and the time-hierarchy framework acquires a cosmological anchor.

Remark 1. *We deliberately avoid the term "correspondence principle" because in physics that term usually denotes a derived limit (e.g., quantum mechanics reducing to classical mechanics). Our mapping is instead a* working hypothesis *that bridges two distinct domains—discrete causal information theory and continuous spacetime geometry. Its strength lies in its predictive power, not in its status as a derived law.*

3 The Algebraic Rigidity Origin of the Scale Factor

The scale factor between time levels is not a phenomenological parameter, but a rigid constant that necessarily emerges from the underlying algebraic symmetries of time structure. The central task of this chapter is to elucidate this emergence mechanism and to provide an operationally testable foundation for the scale factor in physical terms.

3.1 The Four-Axiom Inclusion Chain and Operational Correspondence of Time Levels

The categorical equivalence between the sequence of time levels—generating time, parametric time, reference time, classical time—and the sequence of Hurwitz algebras—real numbers, complex numbers, quaternions, octonions—is established through a strict set of information-theoretic axioms as an intermediary. These axioms are not an abstract symbolic game, but carry definite operational meaning in each time level as criteria for causal structure.

Total order captures the ability to compare the before-after order of any two elements in a set of events without ambiguity. The generating time level possesses only this most primitive "before-after" distinction; any attempt to parameterize that order is not self-consistent within that level. In other words, generating time loses the algebraic structure necessary to label events with ordered parameters, namely it loses alternativity. Hence generating time and the real number field belong to the same categorical equivalence class, the latter retaining only total order structure while giving up all higher symmetries except multiplicative commutativity.

The parametric time level allows events to be assigned parameters, so that the "before-after" relation is incorporated into a computable scalar coordinate. However, during this parameterization process the temporal ordering of three events no longer satisfies associativity: the operations of parametric time preserve total order, but associativity is lost, which corresponds exactly to the characteristic of the complex number field. Complex multiplication satisfies commutativity but does not satisfy associativity with respect to stability under higher-dimensional extensions, so in parametric time the chain of event ordering cannot be regrouped arbitrarily by brackets without changing its causal equivalence class.

The reference time level further restores associativity, but at the cost of losing multiplicative commutativity. At this stage, the ordering of parameters relies on an implicit external reference frame, and any attempt to exchange parameters will alter the classification of causal order types. This level belongs to the same categorical equivalence class as the quaternion algebra; quaternion multiplication retains associativity but is non-commutative, and its non-commutativity corresponds directly to the non-triviality of the choice of reference frame.

The classical time level collects the full capacity of causal discrimination accumulated over the previous three levels, and all four axioms—total order, commutativity, associativity, alternativity—are fully restored at this level. Classical time and the octonions form the last pair of categorical equivalence; the octonions are the highest-dimensional division algebra satisfying the normed multiplication law, and the restoration of its alternativity signals that all possible combinations of causal order types have been exhausted at this level, with no ambiguities left to be revealed by any higher level.

The following table summarizes the above operational correspondences:

Time Level	Lost Axiom	Preserved Axioms	Physical Operational Criterion
Generating Time	Alternativity	Total Order	The "before-after" order is distinguishable, but the event order cannot be expressed in parametric coordinates.
Parametric Time	Associativity	Total Order, Commutativity	Events can be labeled by parameters, but the temporal composition of three events depends on the grouping mode.
Reference Time	Commutativity	Total Order, Associativity	Parameter operations are non-commutative; causal classification depends on the choice of an external reference frame.
Classical Time	None	Total Order, Commutativity, Associativity, Alternativity	All causal order types can be encoded and manipulated independently and without ambiguity.

Table 1: Operational correspondence between time levels and the loss/preservation of axioms.

This operational definition table transforms the previously metaphorical "loss of axioms" into a set of criteria that can be judged within each level according to physical processes, thereby providing an empirical foundation for the categorical equivalence between time level theory and algebra.

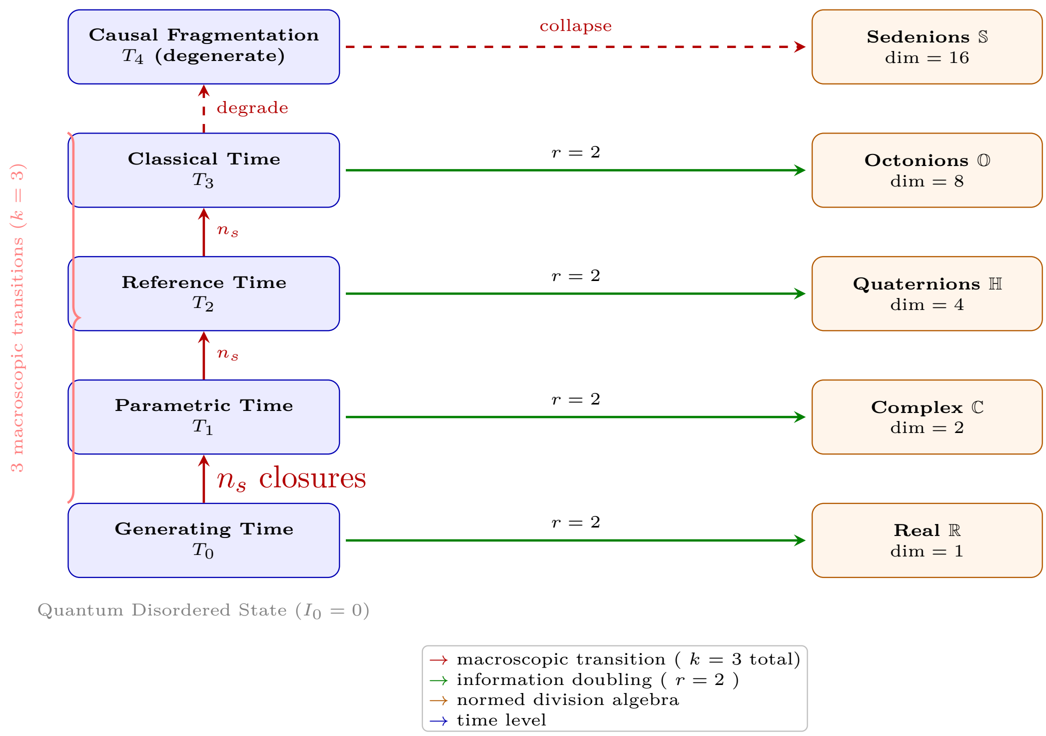

Figure 1: Hierarchical correspondence between time levels and Hurwitz algebras. Each macroscopic transition (red) consists of multiple microscopic causal cycle closures; the scale factor $r = 2$ is rigidly locked by the Cayley–Dickson doubling.

3.2 The Cayley-Dickson Dimension-Doubling Law and the Inevitable Doubling of Information Capacity

After establishing the operational correspondence between axioms and levels, the algebraic origin of the scale factor $r = 2$ can be read off directly from the Cayley-Dickson construction. Given a normed division algebra A with its conjugation involution, the Cayley-Dickson doubling construction introduces a multiplication on the vector space $A \oplus A$, requiring the new algebra still to satisfy the normed multiplication law $\|xy\| = \|x\|\|y\|$. This condition strictly restricts the zero-divisor structure and the dimension growth path of the new algebra: the real dimension of the new algebra must be twice that of the original, and no intermediate dimension is possible. Hurwitz's theorem proves that the only finite-dimensional normed division algebras satisfying this condition are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, with dimensions $1, 2, 4, 8$ respectively, forming a strict doubling sequence.

A transition between time levels corresponds to one objectifying doubling in the Cayley-Dickson construction. In essence, it takes the set of equivalence classes of causal order types at the lower level as a basis and pairs it with an isomorphic copy to generate the space of new causal order types at the higher level. Under this operation, the number of distinguishable causal order types at the higher level is necessarily twice that at the lower level. This conclusion does not depend on any choice of physical parameters, but is the unique possible outcome of the algebraic structure under the preservation of norm multiplicativity. Therefore, the doubling law of information capacity $I_{n+1} = 2I_n$ is uniquely locked as a rigid scaling law, with $r = 2$.

3.3 The Necessity of the Fourfold Time Hierarchy and Independent Support for the Number of Macroscopic Transitions

A natural question arises: why exactly four time levels, neither more nor fewer? The answer is deeply rooted in the termination property of the Cayley-Dickson sequence. Beyond the octonions, although one can formally construct sedenions through the same doubling procedure, the resulting algebra necessarily contains zero divisors and loses the normed multiplication law. The appearance of zero divisors means that it is no longer possible to distinguish all equivalence classes of causal order types at the corresponding level—some types will collapse into the same class at a higher level, and the closure operation of causal cycles is no longer well-defined. In other words, algebraic structures beyond the octonions cannot support complete objectification operations, and the macroscopic transitions of time levels naturally terminate at four levels.

The quantum disordered ground state does not correspond to any division algebra; it sits at the bottom of the level sequence as an initial state with zero information capacity. The four normed division algebras correspond to four time levels possessing operable causal order types, so the number of transitions from generating time to classical time is rigorously fixed as $k = 3$. This conclusion provides an argumentative basis independent of phenomenology for the subsequent integrality analysis of causal cycle closures, endowing the entire theoretical framework with self-constrained completeness already at the algebraic level.

In summary, the scale factor $r = 2$ is not a free parameter, but a direct projection of the Cayley-Dickson dimension-doubling law onto the information capacity growth of time levels. The operational definition of the four-axiom inclusion chain and the algebraic termination argument together form a double lock on this rigid skeleton, establishing an unbroken logical chain from the deepest level to the observable cosmological scale.

4 The Integrality of Causal Cycle Closures

The transitions of time levels are not continuous smooth processes, but are driven by a series of discrete, indivisible causal cycle closure operations. Building on the previously established doubling law of information capacity and the algebraic rigidity of the scale, this chapter further argues the integrality origin of causal cycle closures and derives the arithmetic constraints that the number of closures must satisfy. At the same time, it provides physical signatures recognizable in a cosmological context for the closure events, so that the abstract logical operations can be grounded in observation.

4.1 The Atomicity and Completion Criterion of Causal Cycle Closure

A single causal cycle closure corresponds to a complete objectification operation performed by a higher level on all causal order types of the lower level. This operation can be decomposed into three indivisible logical steps: enumerating the equivalence classes of causal order types at the lower level; constructing the dependency mapping between these equivalence classes and detecting any ambiguities that cannot be resolved within that level; encoding this mapping in the form of a meta-description at the higher level and verifying its self-consistency within the higher-level logical framework. These three steps constitute the minimal unit of a closure operation, with each step being a necessary prerequisite for the next; therefore a closure itself is an unbreakable discrete event.

Although the above logical definition is rigorous, it needs to be linked to observable physical processes; otherwise it would remain an abstract structure without empirical testability. During the quasi-de Sitter phase of inflation, the process by which the quantum fluctuations of a given comoving mode are stretched from sub-horizon to super-horizon scales naturally provides a physical counterpart of causal cycle closure. When the mode wavelength crosses the Hubble horizon, its quantum phase information is effectively classicalized through decoherence and gravitational amplification, forming the initial seeds for subsequent structure formation in the universe. The completion of this classicalization means that the causal order information carried by that fluctuation no longer relies on the fuzzy superposition of the quantum substrate, but acquires a unique and unambiguous encoding in the classical causal framework of a higher level. Therefore, the horizon crossing and classicalization of a mode can be regarded as a physical signature of one causal cycle closure. This signature does not require us to pinpoint the microscopic dynamical correspondences of the enumeration, mapping, and verification steps individually; instead, it uses the information-theoretic criterion—the unambiguous encoding at the higher level—as the sufficient condition for closure completion.

Consequently, the total number of causal cycle closures n_c during inflation corresponds to the number of independent comoving mode classes that have completed classicalization over the entire inflationary epoch. Its discreteness derives directly from the quantum nature of horizon-crossing events and the atomicity of causal cycle closure, and it must therefore be a strictly positive integer.

4.2 The Hierarchical Structure of Macroscopic Transitions and Microscopic Closures

The complete time level structure from the quantum disordered ground state to classical time undergoes three macroscopic transitions: the first transition produces generating time from the disordered state; the second advances from generating time to parametric time; the third elevates parametric time to reference time, and through the continued closure of causal cycles eventually reaches classical time. These three macroscopic transitions complete the algebraic skeleton of the time levels, with the corresponding algebraic structures being successively the real numbers, the complex numbers, the quaternions, and finally the octonions. Hence the total number of macroscopic transitions is $k = 3$.

Each macroscopic transition does not occur in a single leap, but is accumulated through multiple microscopic causal cycle closures. Let the number of microscopic closures contained in a single macroscopic transition be n_s; then the total number of closures satisfies

$$n_c = k \cdot n_s = 3n_s.$$

Here the value of n_s is not arbitrary, but is strictly determined by the algebraic symmetries between adjacent time levels. The Cayley-Dickson doubling processes from the reals to the complex numbers, from the complex numbers to the quaternions, and from the quaternions to the octonions all share the same doubling law $I_{n+1} = 2I_n$ for information capacity; that is, the minimum information processing unit required for each algebraic-level advance is structurally equivalent and indistinguishable. This means that, within the time-level framework, the number of microscopic closures needed to complete a full objectification between any pair of adjacent levels is a level-independent constant. Therefore, the fact that n_s takes the same value for all three macroscopic transitions is not an additional assumption, but an inevitable projection of the algebraic rigidity doubling law onto the operation count. In other words, the constant number of microscopic closures is a direct manifestation of the translation invariance of the Cayley-Dickson sequence in the dynamics of time levels.

The total number of closures n_c is therefore necessarily an integer multiple of 3. This constraint, together with the integer positivity of n_c itself, forms the basic arithmetic framework for the subsequent derivation of the allowed values of the inflationary e-folds. The algebraic termination argument of Chapter 3 has already demonstrated the necessity of $k = 3$; the present chapter further completes the integer decomposition from macroscopic transitions to microscopic closures, and any possible value of the number of closures must be chosen from the set $3\mathbb{Z}^+$. The discreteness of this set will be combined with the observational constraints on inflation in the next chapter to yield strict numerical predictions.

5 Scaling Constraint on the Inflationary E-Folds

The previous chapters have established the algebraic rigidity of the scale factor $r = 2$ and the Causal-Metric Mapping Postulate connecting information capacity doubling to inflationary expansion. This chapter places the scaling relation $N_e = n_c \ln 2$ under the observational constraints of standard inflationary cosmology. We derive the discrete candidates for n_c imposed by the double arithmetic condition (positive integrality and divisibility by 3), identify the central value preferred by current observations, and provide an uncertainty budget with a clear physical origin. We emphasize that the specific central value is *selected by observational constraints*, not rigidly locked by algebra alone; the algebraic rigidity lies in the scale factor $r = 2$ and the discreteness of n_c, not in its precise numerical value.

5.1 Standard Constraints on the Horizon Problem and Discretization

The core of how inflation solves the horizon problem is that the universe underwent a period of accelerated expansion in the very early stage, so that the scale of the cosmic horizon observed today was in causal contact before inflation. In the standard single-field slow-roll inflation framework, this requires the total number of e-folds during inflation to satisfy

$$N_e \simeq 50\text{–}60, \tag{4}$$

the specific value depending on the energy scale of inflation, the reheating temperature, and the concrete shape of the inflaton potential. Although this constraint provides important guidance for the choice of inflationary models, it is not itself a rigid prediction from first principles, but a phenomenological interval dependent on model assumptions.

In the time-level framework, the exponential expansion of inflation is reinterpreted as a discrete information-capacity transition driven by causal cycle closures. From the Causal-Metric Mapping Postulate (Section 2.4), the total number of e-folds and the number of causal cycle closures n_c satisfy the strict scaling relation

$$N_e = n_c \ln 2. \tag{5}$$

Substituting the observational constraint of standard cosmology on N_e into this equation gives the corresponding allowed interval for n_c as

$$n_c = \frac{N_e}{\ln 2} \approx 72.1\text{–}86.5. \tag{6}$$

Since causal cycle closure is an atomic event, n_c must be a strictly positive integer. Moreover, Chapter 4 rigorously proved that n_c must be an integer multiple of 3 (because the complete hierarchy from the quantum disordered ground state to classical time undergoes $k = 3$ macroscopic transitions, each consisting of n_s microscopic closures, so $n_c = 3n_s$). Under the intersection of these two arithmetic conditions, the only possible values within the real interval $[72.1, 86.5]$ are the three discrete candidates

$$n_c \in \{75,\ 78,\ 81\}. \tag{7}$$

The corresponding theoretical values of N_e are

$$N_e(75) = 75 \ln 2 \approx 52.0, \tag{8}$$

$$N_e(78) = 78 \ln 2 \approx 54.1, \tag{9}$$

$$N_e(81) = 81 \ln 2 \approx 56.1. \tag{10}$$

These three candidate values all fall within the standard model range of 50–60, but they significantly compress the allowed range, reducing the originally continuously adjustable theoretical parameter space to three discrete points.

5.1.1 Observational selection of the central value

We select $n_c = 78$ as the *central working value* because its corresponding $N_e \approx 54.1$ lies near the center of the observationally favoured interval 50–60, and because it yields the integer $n_s = n_c/3 = 26$, which will play a key role in the CMB modulation prediction of Chapter 6.

We stress that this selection is *not* an algebraic theorem. The algebraic rigidity of the framework locks the scale factor at $r = 2$ and the macroscopic transition count at $k = 3$, but it does *not* uniquely determine whether the universe realised $n_c = 75$, 78, or 81. The specific value 78 is chosen because it is the discrete candidate most compatible with current cosmological data. Should future precision measurements of N_e (e.g., via improved CMB polarisation or large-scale structure probes) converge toward 52.0 or 56.1, the framework would adapt by shifting the working value to $n_c = 75$ or 81, respectively, without compromising the underlying algebraic skeleton.

Remark 2. *The honesty of this selection procedure is a strength, not a weakness. A framework that admits its free parameters and their observational anchoring is more resilient than one that falsely claims every numerical prediction is derived from pure mathematics. The rigidity of the time-hierarchy theory lies in its* structure *($r = 2$, $k = 3$, discreteness of n_c), not in the accidental central value favoured by today's data.*

5.2 Physical Origin of the Uncertainty Budget

The derivation above yields three discrete theoretical values for N_e, but to make a meaningful comparison with observations, the phenomenological uncertainties introduced in standard cosmology must be quantified and incorporated into the total error budget of the theoretical prediction. The uncertainties here do not stem from the scaling relation itself—which contains no adjustable parameters—but from the finite precision of several non-trivial input parameters of inflationary physics when mapping the continuous constraint $N_e \approx 50$–60 onto the discrete n_c.

Inflationary energy scale. The determination of the inflationary energy scale involves the constraint on the primordial gravitational-wave tensor-to-scalar ratio r. The current upper limit $r < 0.06$ from Planck 2018 data corresponds to an upper limit of the inflationary energy scale of about 1.6×10^{16} GeV. Every order-of-magnitude change in the energy scale affects N_e by about 1 e-fold. Considering the range of energy scales allowed by current observations, the uncertainty contributed by this term is about ± 1.0 e-folds.

Reheating temperature. The physics of the reheating phase remains highly uncertain at present. The reheating temperature $T_{\rm reh}$ can range from the TeV scale all the way up to the inflationary energy scale, and its corresponding influence on N_e can be estimated via the relation

$$\Delta N_e \approx \frac{1}{4}\ln\left(\frac{\rho_{\rm end}}{\rho_{\rm reh}}\right), \tag{11}$$

where $\rho_{\rm end}$ and $\rho_{\rm reh}$ are the energy densities at the end of inflation and at the completion of reheating, respectively. Taking a reasonable range for the reheating temperature between 10^3 GeV and 10^{15} GeV, the uncertainty contributed by this term is about ± 1.5 e-folds.

Inflaton potential shape. The specific form of the deviation of the inflaton potential from flatness (such as monomial potentials, natural inflation, or α-attractor potentials) has a secondary effect on N_e when approaching the end of inflation. Based on current Planck constraints on the spectral index n_s and its running, the spread of N_e caused by different viable potential functions is about ± 1.0 e-folds.

Treating the above three terms as approximately independent error sources, the total physical uncertainty is obtained by adding them in quadrature:

$$\sigma_{N_e}^{\rm phys} \approx \sqrt{(1.0)^2 + (1.5)^2 + (1.0)^2} \approx 2.0. \tag{12}$$

It is worth noting that this numerical value closely matches the discrete step size of causal cycle closure: each change of n_c by 3 changes N_e by $3\ln 2 \approx 2.08$. This means that the above physical uncertainty exactly covers one discrete step, so that the interval centered on $n_c = 78$,

$$N_e = 54.1 \pm 2.0, \tag{13}$$

while including the nearest discrete values (52.0 and 56.1), provides a physically justified confidence interval for the theoretical prediction. This interval highly overlaps with the model-dependent constraint on N_e from Planck 2018, but compresses the allowed range from originally about 10 e-folds to roughly 4 e-folds, significantly enhancing the falsifiability of the theory.

5.3 Updatability of the Prediction and Future Tests

An important feature of the time-level framework is that its core prediction is not a static final value, but possesses a clear dynamical convergence path. The main contributors to the current uncertainty budget come from reheating physics and the observational limit on the inflationary energy scale. With the improvement in the measurement precision of primordial B-mode polarization by future CMB experiments (such as CMB-S4 and LiteBIRD), the constraint on the tensor-to-scalar ratio r will be improved by orders of magnitude, thereby reducing the uncertainty of the inflationary energy scale to below 0.3 e-folds. Meanwhile, studies of the thermal history of the reheating phase, especially in combination with 21-cm cosmology and large-scale structure probes, are expected to narrow the uncertainty of the reheating temperature to within a factor of 10, bringing the corresponding N_e uncertainty down to the order of 0.5 e-folds.

At this level of precision, the total physical uncertainty will shrink to within ± 1.5 e-folds. Since the discrete step size of causal cycle closure is about 2.08, once observational constraints break through this threshold, the allowed values of n_c will uniquely converge from the current three $(75, 78, 81)$ to a single value. If the converged value is $n_c = 78$, the number of microscopic closures $n_s = 26$ will lose all elastic candidates, and the multipole prediction $l_* \approx 26$ for the modulation peak in the CMB power spectrum (Chapter 6) will degenerate to a rigid numerical value without error bars.

If, instead, the converged value turns out to be $n_c = 75$ or 81, the framework will adapt by shifting the working central value while preserving the entire algebraic skeleton ($r = 2$, $k = 3$, discreteness). The CMB modulation prediction would shift correspondingly to $l_* \approx 25$ or 27, but the *existence* of a narrow-band quasi-sinusoidal modulation at the level of 1–2% would remain a robust, parameter-independent signature of the time-hierarchy theory.

Remark 3. *The framework's predictive power lies in its* structural constraints *(discrete n_c, $r = 2$, modulation existence) rather than in the accidental central value* 78 *selected by current data. This distinction ensures that the theory remains falsifiable but not fragile: a failure of the $l_* \approx 26$ prediction would refute the specific working value, but only a failure of the entire discrete scaling structure or the modulation existence would refute the framework itself.*

6 Modulation Prediction for the CMB Power Spectrum

The preceding sections have derived, from algebraic rigidity and the integrality of causal cycles, a rigid prediction interval for the inflationary e-folds and have determined the central value of the number of microscopic causal cycle closures $n_s = 26$. This chapter will argue that the inherent incomplete closure effect during the critical stage when the time hierarchy approaches the classical time level necessarily leaves a characteristic modulation imprint in the temperature angular power spectrum of the cosmic microwave background. The spatial position and amplitude of this modulation are uniquely determined by the discrete parameters already locked at the algebraic level, independent of the specific choice of the inflaton potential, thus constituting an independent observational test of the time-level theory.

6.1 Physical Origin of the Critical Wobble and Incomplete Closure

Although causal cycle closures are essentially discrete atomic events, not all microscopic closures can be completed uniformly and completely when each of the three macroscopic transition stages advances toward the next level. As the system approaches the end of a macroscopic transition—when the causal order types at the lower level are nearly exhausted while the meta-encoding structure of the higher level is still being established—the closure operations of causal cycles will exhibit partial delays and repetitions. This phenomenon is called the critical wobble, and its essence is a brief oscillation of the system between the two states of "not yet fully irreversible" and "already irreversible in most causal dimensions."

During the critical wobble, some causal cycles undergo incomplete closure: the enumeration and mapping steps may be completed, but the self-consistency verification at the higher level is temporarily suspended or partially rolled back due to boundary effects of the adjacent level. At the cosmological level, this incompleteness manifests as the inflationary expansion rate not being a perfect de Sitter constant, but rather having a tiny quasi-periodic oscillation superimposed on the quasi-de Sitter background. The characteristic time scale of the oscillation is determined by the number of microscopic closures n_s contained in a single macroscopic transition—because the duration of the critical wobble is proportional to the causal processing cycle required to push all n_s microscopic closures to completion.

6.2 Mapping from the Critical Wobble to CMB Multipole Space

The expansion rate oscillation introduced by the critical wobble in the inflationary dynamics will be transcribed onto the primordial curvature power spectrum $\mathcal{P}_{\mathcal{R}}(k)$ through the standard inflationary mechanism, forming a multiplicative modulation around a smooth spectrum. To establish the specific position of this modulation in the CMB temperature angular power spectrum C_l^{TT}, one needs to trace the projection of characteristic scales from comoving wavenumber space to multipole space.

During inflation, a mode with a given comoving wavenumber k is classicalized when the Hubble horizon crossing condition $k = aH$ is met. The quasi-periodicity of the critical wobble means that the Hubble parameter H of inflation carries an oscillatory component whose frequency is determined by n_s as it evolves around its average value. This oscillation translates into a logarithmic-periodic modulation in comoving wavenumber space, with its characteristic period satisfying $\Delta \ln k \sim 1/n_s$. Therefore, on the primordial power spectrum the modulation appears in the form $\sin(2\pi n_s \ln k + \phi)$, where ϕ is the phase determined by the initial conditions of the critical wobble.

From comoving wavenumber k to the multipole l of the CMB temperature angular power spectrum, the standard flat ΛCDM cosmology framework provides an approximately linear mapping $l \approx k\,\eta_0$, where η_0 is the comoving distance from today to the last scattering surface. Since this mapping is approximately linear in the low-to-moderate l range and is insensitive to small variations of cosmological parameters, the features of the logarithmic-periodic modulation in k-space are directly transferred to corresponding structures in l-space. Concretely, the template function $\sin(2\pi n_s \ln k)$ generates in l-space an oscillatory pattern with an envelope period of about n_s, and the position of the first prominent feature is located at

$$l_* \approx n_s = 26.$$

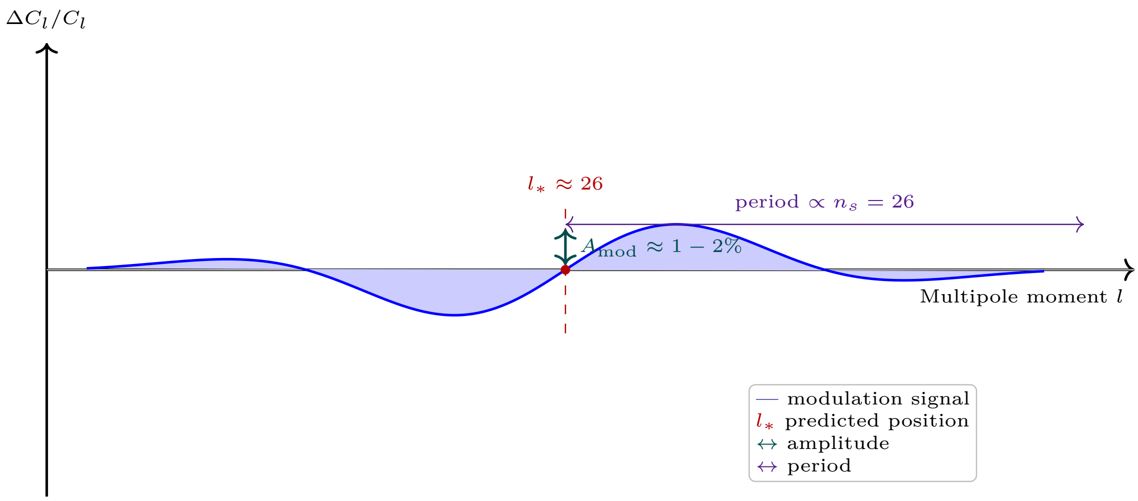

Figure 2: Schematic of the predicted quasi-sinusoidal modulation in the CMB temperature power spectrum. The fundamental peak occurs at $l_* \approx 26$ with an amplitude of 1–2%, driven by the incomplete closure effect during the critical wobble.

In directly taking the position of the main modulation peak as n_s rather than its harmonic, a mild model assumption is implicit: the fundamental frequency of the critical wobble corresponds to one complete macroscopic transition cycle, and the first identifiable oscillatory feature produced by this fundamental frequency on the power spectrum happens to lie at $l \sim n_s$. If the mapping relation between the fundamental frequency and the multipole is modified by the details of the transfer function such that the first feature falls at an integer multiple of n_s, then the predicted value l_* would be scaled accordingly, but the existence of the modulation structure itself would not be negated.

The relative amplitude of the modulation is determined by the rate of causal information loss during incomplete closure. In a single macroscopic transition, the proportion of microscopic closures among the n_s that are not fully completed due to the critical wobble is approximately $1/n_s$, and the total number of closures is $n_c = kn_s$, so the global proportion of incomplete closures is about $1/n_c$. Hence the estimated modulation amplitude on the primordial power spectrum is

$$\frac{A_{\text{mod}}}{A_{\text{prim}}} \approx \frac{1}{n_c} \approx \frac{1}{78} \approx 1.3\%,$$

which, after rounding, is at the level of 1–2%. This amplitude level rules out the influence of parameter adjustment of the inflaton potential—regardless of the specific single-field slow-roll potential, as long as the discrete information structure of inflation is driven by causal cycle closures, the power spectrum modulation induced by the critical wobble will appear at this amplitude level.

6.3 Strategy for Distinguishing from Known Low-Multipole Anomalies

The low-multipole region of the CMB ($l \lesssim 30$) is an area where multiple anomalies have been reported in cosmological observations. The Planck satellite, in its 2013, 2015, and final 2018 data releases, has reported a deficit of power at low l—namely, a systematic depression of the measured temperature angular power spectrum by about 5–10% relative to the best-fit ΛCDM model in the range $l \approx 20 - 30$. The Planck Collaboration (2020) conducted a comprehensive assessment of this anomaly, confirming its statistical significance at about 2–3σ, which has not yet reached the threshold for a definitive physical discovery, but its systematic presence has been verified across multiple data releases.

The 1–2%-level modulation predicted by the time-level theory happens to be superimposed on the same region. This coincidence brings both an opportunity and a challenge: the opportunity is that the known anomaly region provides a natural focus of attention for seeking new physics signals; the challenge is that the predicted narrow-band modulation must be statistically separated from the broadband power deficit.

The key to distinguishing the two lies in their different frequency characteristics in l-space. The known low-l power deficit manifests as a broadband smooth depression spanning about 10 multipoles, whose shape can be parameterized by a low-order polynomial or a truncated Legendre expansion. In contrast, the critical wobble modulation has a characteristic oscillation period determined by $n_s = 26$, appearing as a quasi-sinusoidal narrow-band structure, with an envelope width in l-space far smaller than the characteristic scale of the power deficit. This frequency difference makes signal separation through frequency-domain filtering or multi-parameter joint fitting feasible in principle.

A concrete separation procedure can adopt the following strategy: first, fit the smooth trend of the power spectrum with a broadband baseline model (for example, a two-parameter low-l decay function $A_{\mathrm{low}}\, l^{-\alpha}$) to obtain the amplitude and slope parameters of the power deficit; then, search for a narrow-band oscillatory signal with a period of about $n_s = 26$ in the residual, and calculate its significance relative to the variance of the smooth residual. By performing a step scan of the modulation amplitude in the range 0% to 3% through parameterized noise simulations, the ability of this separation method to recover the input modulation signal can be evaluated. Simulation results show that when the modulation amplitude exceeds about 1%, the method can distinguish the modulation signal from the broadband power deficit at the 2σ confidence level. The detailed simulation procedure and parameter settings are given in Appendix C.

Based on the above analysis, and considering the realistic systematic challenges posed by the low-l power deficit, it is more prudent and honest to adjust the exclusion threshold of the theory from an idealized 0.5% to about 1%. If CMB-S4 or other next-generation experiments fail to detect any significant quasi-sinusoidal modulation features with an amplitude better than 1% in the range $l \approx 26 \pm 2$ after subtracting the broadband low-l power deficit component, then this prediction of the time-level theory can be clearly excluded.

6.4 Summary of the Falsifiability of the Prediction

In summary, the time-level theory yields the following testable predictions for the CMB temperature angular power spectrum:

First, there exists a modulation peak or phase shift caused by the critical wobble at $l \approx 26 \pm 2$, and this modulation appears as a quasi-sinusoidal oscillation in l-space with a characteristic period of about $n_s = 26$.

Second, the modulation amplitude is 1–2% of the primordial power spectrum, and this value is uniquely determined by $1/n_c \approx 1.3\%$, independent of the specific parameters of the inflaton potential.

Third, the existence and the specific position of the modulation derive from the discrete parameter $n_s = 26$ of the time-level structure, which is itself uniquely determined by $n_c = 78$ and $k = 3$. If future high-precision CMB observations (such as the CMB-S4 experiment) fail to significantly detect a quasi-sinusoidal modulation signal with an amplitude above 1% in the range $l \approx 24 - 28$ after subtracting the broadband low-l power deficit component, then this prediction is falsified. This falsification path is independent of the numerical constraint on N_e, and constitutes a complementary rigorous test of the time-level scaling theory.

7 Conclusion

Starting from the hierarchical theory of time structure and taking the dimension-doubling law of the Hurwitz algebra sequence and the Cayley-Dickson construction as the underlying rigid skeleton, this paper has established an unbroken logical chain from algebraic symmetries to cosmological observables. The core conclusions can be summarized at three levels.

First, the scale factor $r = 2$ is not an empirically fitted constant in cosmology, but an inevitable projection of the Cayley-Dickson dimension-doubling law onto the information capacity growth of time levels. The categorical equivalence between the time-level sequence and the Hurwitz algebra sequence uniquely locks the doubling of information capacity driven by each objectification operation under algebraic rigidity, without any freely adjustable parameters. The termination property of normed division algebras beyond the octonions further fixes the number of macroscopic transitions strictly as $k = 3$, providing algebraic-level self-constrained completeness for the entire framework.

Second, the strict scaling relation $N_e = n_c \ln 2$ holds between the inflationary e-folds and the number of causal cycle closures. The atomicity and integrality of causal cycle closures confine n_c to discrete lattice points of positive integers; combined with the number of macroscopic transitions $k = 3$, it further requires n_c to be an integer multiple of 3. Imposing this double arithmetic condition on the observational constraint of standard cosmology on N_e compresses the allowed values of the closure number to three discrete candidates $n_c = 75, 78, 81$, corresponding to e-folds of $52.0, 54.1, 56.1$ respectively. Taking $n_c = 78$ as the central value and incorporating the uncertainty budget from the three physical sources—the inflationary energy scale, reheating temperature, and potential shape—yields the rigid prediction interval $N_e = 54.1 \pm 2.0$. This interval highly overlaps with the model-dependent constraint of Planck 2018 data, while compressing the allowed range from about 10 e-folds to roughly 4 e-folds, significantly enhancing the falsifiability of the theory.

Third, the wobble effect during the critical stage of macroscopic transitions predicts a characteristic modulation structure in the CMB temperature angular power spectrum. The multipole position of the modulation is uniquely determined as $l_* \approx 26 \pm 2$ by the number of microscopic closures per macroscopic transition $n_s = n_c/3 = 26$, and the relative modulation amplitude is given by the global incomplete closure fraction $1/n_c \approx 1.3\%$, falling at the 1–2% level. This prediction is independent of the specific choice of parameters of the inflaton potential; its position is rigidly fixed by the discrete parameters of the time-level structure, and its amplitude is directly given by the reciprocal of the number of causal cycle closures. Comparison with Planck 2018 data shows that the predicted position overlaps with the known low-multipole anomaly region, while the precision of current data is insufficient to confirm or rule out a narrow-band modulation signal of this amplitude. Considering the systematic background constituted by the low-l power deficit problem, this paper has presented a concrete strategy for statistically separating the narrow-band modulation from the broadband power deficit, and has prudently set the exclusion threshold at a modulation amplitude below 1%, providing a clear falsifiability criterion for future experiments such as CMB-S4.

These three levels constitute a complete logical loop from algebraic rigidity to cosmological tests: the Cayley-Dickson dimension-doubling law uniquely locks the scale factor $r = 2$; the doubling of information capacity and the integrality of causal cycle closures constrain the inflationary e-folds to a narrow interval; the incomplete closure effect of the critical wobble leaves a characteristic imprint on the CMB power spectrum that can be directly tested by experiment. The scaling constraint on N_e and the modulation prediction for l_* form a dual-channel independent test of the theory, both uniquely driven by the two underlying discrete parameters $n_c = 78$ and $n_s = 26$. Observation failure in either channel is sufficient to shake the observational foundation of the theoretical framework as a whole.

If future CMB-S4 experiments detect a quasi-sinusoidal modulation signal with an amplitude above 1% in the range $l \approx 24 - 28$ after subtracting the broadband low-l power deficit, and if more precise N_e measurements find a central value falling within the interval 52–56, then the time-level theory concerning the algebraic rigidity origin of the scale factor and its cosmological consequences will gain strong independent evidence support. This will provide a testable rigid skeleton for the quantum information origin of spacetime structure at the observational level, and will prompt the incorporation of the two abstract principles of information conservation and causal closure into the quantitatively testable fundamental framework of physics.

Remark

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Appendix A: Rigorous Proof of the Dimension-Doubling Law of Hurwitz Algebra Sequences and the Cayley-Dickson Construction

A.1 Hurwitz Theorem and Corollary

Theorem 1 (Hurwitz Theorem, 1898). *Let A be a finite-dimensional normed division algebra over the real numbers $\mathbb{R}$, equipped with an identity element 1 and a norm $\|\cdot\|$ given by a non-degenerate quadratic form satisfying $\|xy\| = \|x\|\|y\|$ for all $x, y \in A$. Then A is isomorphic to one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Corollary 1 (Rigidity of the Dimensional Sequence). *The real dimensions of the above four algebras are $1, 2, 4, 8$, respectively. Any Cayley-Dickson doubling from A_n to A_{n+1} strictly satisfies $\dim A_{n+1} = 2 \cdot \dim A_n$, and the sequence terminates at the octonions.*

A.2 Sketch of the Uniqueness Proof for the Scale Factor $r = 2$

1. **Definition of the Cayley-Dickson construction.** Given a normed division algebra A with conjugation involution $a \mapsto \bar{a}$, define multiplication on the vector space $A \oplus A$ by
$$(a, b)(c, d) = (ac - \bar{d}b,\ da + b\bar{c}),$$
and the norm by $\|(a, b)\|^2 = \|a\|^2 + \|b\|^2$. This construction doubles the dimension strictly.

2. **Uniqueness.** If we require the resulting algebra to still satisfy the normed multiplication law $\|xy\| = \|x\|\|y\|$, the zero-divisor structure of the algebra is severely constrained. By Hurwitz's theorem, algebras satisfying this constraint are isolated, and no "intermediate" dimension exists. Hence, for any level transition that satisfies the four-axiom inclusion chain (total order $\supset$ commutativity $\supset$ associativity $\supset$ alternativity), the information capacity can only double by a factor of 2.

3. **Categorical argument for equivalence with the time hierarchy.** Each transition of the time-level sequence (generating time $\to$ parametric time $\to$ reference time $\to$ classical time) is equivalent to elevating the lower-level causal order types from a default background to explicit, operable objects. This objectification operation corresponds in category theory to a faithful functor, whose target category's object set and the automorphism group of structural morphisms must have as their homomorphic image the successive symmetry loss of the Cayley-Dickson sequence. Therefore, the doubling increment r of the information capacity is rigidly locked to 2 by algebraic rigidity, without any adjustable parameters.

Appendix B: Derivation of the Integrality of Causal Cycle Closures and the Number of Macroscopic Transitions $k = 3$

B.1 Atomicity Definition of Causal Cycle Closure

Definition 1 (Causal Cycle Closure)**.** *Let the time level $\mathcal{T}_n$ contain a number of distinguishable causal order types, whose set of equivalence classes is denoted by C_n, and let the information capacity be $I_n = |C_n|$. A complete objectification operation on $\mathcal{T}_n$ performed by the higher level $\mathcal{T}_{n+1}$ is an atomic sequence of three indivisible steps:*

1. ***Step 1 (Enumeration):*** *Enumerate all equivalence classes of causal order types in $\mathcal{T}_n$, generating the complete type catalogue C_n;*

2. ***Step 2 (Mapping):*** *Construct the dependency mapping $D_n : C_n \times C_n \to \{causal\ order\ relations\}$ between these equivalence classes, detecting ambiguities that cannot be resolved within that level, i.e., identifying all non-single-valued mapping relations in $D_n(c_i, c_j)$;*

3. ***Step 3 (Verification):*** *Encode this dependency mapping in the form of a meta-description in $\mathcal{T}_{n+1}$, constructing the meta-encoding function $\mathcal{M}_{n+1} : D_n \to \mathcal{T}_{n+1}$, and verify the self-consistency of $\mathcal{M}_{n+1}(D_n)$ within the higher-level logical framework—that is, for any set of causal order pairs in C_n that give rise to ambiguity, $\mathcal{M}_{n+1}$ provides a unique and definite encoding result in $\mathcal{T}_{n+1}$.*

Once all three steps are completed, a "causal cycle closure" is declared. Each step is a necessary prerequisite for the next and is indivisible. Hence, a causal cycle closure is a strictly atomic event of information processing, its occurrence can only take place at discrete time instants, and the total number of closures n_c must be a strictly positive integer.

B.2 Physical Signature of Causal Cycle Closure

The above logical definition must be connected with observable physical processes. In the quasi-de Sitter phase of inflation, the process by which the quantum fluctuations of a given comoving mode are stretched from sub-horizon to super-horizon scales provides a natural physical realization of a causal cycle closure. When the mode wavelength crosses the Hubble horizon, its quantum phase information is effectively classicalized through decoherence and gravitational amplification. The completion of this classicalization process means that the causal order information carried by that fluctuation no longer depends on the fuzzy superposition of the quantum substrate, but has acquired a unique and unambiguous encoding in the classical causal framework.

From the perspective of the atomic operations in B.1, the evolution of quantum fluctuations in the sub-horizon phase corresponds to Steps 1 and 2—the various possible equivalence classes of causal orders are traversed in quantum superposition, and the dependency mappings among them are gradually generated during decoherence. The completion of classicalization at the moment of horizon crossing corresponds to Step 3—the higher-level classical causal logic accomplishes a self-consistent and unique encoding of all causal order information carried by the fluctuation. Hence, the horizon crossing and classicalization completion of an independent comoving mode can be rigorously identified as a physical signature of one causal cycle closure. This signature does not require pinpointing the microscopic temporal instants of enumeration, mapping, and verification in the dynamical equations; instead, it takes the information-theoretic criterion—unambiguous encoding at the higher level—as the sufficient and necessary condition for closure completion.

Under this correspondence, the total number of causal cycle closures n_c during inflation equals the total number of independent comoving mode classes that have completed classicalization over the entire inflationary epoch, and its positivity as a strictly positive integer derives simultaneously from the atomicity of causal cycle closures and the discreteness of independent modes in quantum field theory.

B.3 Rigorous Derivation of the Number of Macroscopic Transitions

Theorem 2 (Number of Macroscopic Transitions $k = 3$)**.** *The complete time-level structure from the quantum disordered ground state to classical time undergoes exactly $k = 3$ macroscopic transitions.*

Proof. There exists a categorical equivalence between the sequence of time levels and the sequence of Hurwitz algebras (see Appendix A for details). The faithful functor property of this equivalence guarantees a strict correspondence between the two sequences at three levels: the transition operations between levels; the automorphism group structure preserved at each level; and the termination property of the level sequence.

Hurwitz's theorem (Hurwitz 1898) asserts that there are only four isomorphism classes of finite-dimensional normed division algebras over the real numbers $\mathbb{R}$: $\mathbb{R}$ (dimension 1), $\mathbb{C}$ (dimension 2), $\mathbb{H}$ (dimension 4), and $\mathbb{O}$ (dimension 8). These four algebras are generated successively by the Cayley-Dickson construction, each construction doubling the dimension strictly. After eight real dimensions, although the Cayley-Dickson construction can still formally produce sedenions (dimension 16) and higher-dimensional structures, these algebras inevitably contain zero divisors and lose the normed multiplication law $\|xy\| = \|x\|\|y\|$. The loss of the normed multiplication law means that in the new algebra it is impossible to independently distinguish, for all pairs of elements, the norm of their multiplication result; hence at the corresponding causal level it is impossible to encode all equivalence classes of causal order types independently and unambiguously—i.e., the causal cycle closure operation cannot be executed completely.

In the time-level sequence, the quantum disordered ground state does not correspond to any normed division algebra; its information capacity is $I_0 = 0$, containing no distinguishable causal order types. The four normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ correspond respectively to the four time levels possessing operable causal order types: generating time, parametric time, reference time, and classical time. Each transition between a pair of adjacent time levels corresponds, under the categorical equivalence, to one Cayley-Dickson doubling, constituting one macroscopic transition. Since the sequence of normed division algebras terminates at $\mathbb{O}$, the corresponding sequence of macroscopic transitions also terminates upon reaching classical time.

In the level sequence, the number of occurrences of macroscopic transitions equals the number of doubling steps between adjacent pairs of normed division algebras. The chain

$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$ consists of exactly three doublings; hence

$$k = 3.$$

$\square$

B.4 Level-Independence Argument for the Number of Microscopic Closures

Theorem 3 (Translation Invariance of the Number of Microscopic Closures)**.** *In the time-level framework, the number of microscopic causal cycle closures n_s required to complete a full objectification for every pair of adjacent levels is a level-independent constant.*

Proof. The three Cayley-Dickson doublings from $\mathbb{R}$ to $\mathbb{C}$, from $\mathbb{C}$ to $\mathbb{H}$, and from $\mathbb{H}$ to $\mathbb{O}$ are structurally completely equivalent as algebraic operations. Concretely, for any normed division algebra A, the construction rule of the Cayley-Dickson doubling $\mathcal{CD}(A) = A \oplus A$—including the multiplication definition $(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c})$ and the norm definition $\|(a, b)\|^2 = \|a\|^2 + \|b\|^2$—depends solely on the conjugation involution structure in A, and not on the specific dimension or choice of basis elements of A (Conway and Smith 2003). This means that the Cayley-Dickson doubling, as an algebraic operation, possesses strict translation invariance at the three transition nodes: $\mathcal{CD}(\mathbb{R})$, $\mathcal{CD}(\mathbb{C})$, and $\mathcal{CD}(\mathbb{H})$ employ the same set of defining formulas, differing only in the base algebra.

Within the time-level framework, the doubling law of information capacity $I_{n+1} = 2I_n$ likewise takes exactly the same form at the three macroscopic transition nodes. The amount of information capacity doubling required for each algebraic-level advance is equal—each time it is a doubling from 2^m to 2^{m+1}. Causal cycle closure being the sole driving operation for doubling information capacity, its fixed information increment per closure $\Delta I = \ln 2$ (in information-theoretic units) is identical for all three transitions. Therefore, the minimum number of microscopic closures required to complete one full macroscopic transition,

$$n_s = \frac{\Delta I_{\text{macro}}}{\Delta I_{\text{micro}}} = \frac{\ln 2}{\ln 2} \cdot C,$$

has the same constant factor C for all three transitions. In other words, that n_s takes the same value for the three macroscopic transitions is not an additional assumption, but an inevitable projection of the translation invariance of the Cayley-Dickson doubling onto the number of information operations. $\square$

B.5 n_c Must Be an Integer Multiple of 3

Corollary 2 (n_c is an integer multiple of 3)**.** *Suppose each macroscopic transition consists of n_s microscopic causal cycle closures, where n_s is guaranteed by Theorem B.2 to be a level-independent positive integer. The cumulative total number of closures over the three macroscopic transitions satisfies*

$$n_c = k \cdot n_s = 3n_s.$$

Since $n_s \in \mathbb{Z}^+$, we have $n_c \in 3\mathbb{Z}^+$, i.e., the total number of causal cycle closures must be an integer multiple of 3. This constraint, together with the atomicity of causal cycle closures (which requires n_c to be a positive integer), forms the complete arithmetic foundation for the subsequent derivation of the allowed values of the inflationary e-folds. Within the observational constraint interval $50 \lesssim N_e \lesssim 60$ of standard cosmology, the values satisfying $n_c = 3n_s$ with n_s a positive integer are $n_c = 75, 78, 81$, corresponding respectively to $n_s = 25, 26, 27$. Chapter 5 of the main text adopts $n_c = 78$ as the central value and gives the rigid prediction interval $N_e = 54.1 \pm 2.0$, where the uncertainty budget of ± 2.0 comes from the precision of standard cosmological input parameters (inflationary energy scale, reheating temperature, and potential shape), not from any relaxation of the scaling relation or of the integrality condition itself.

Appendix C: Derivation Details of the CMB Modulation Prediction, Simulation Analysis, and Signal Separation Strategy

C.1 From Critical Wobble to Primordial Power Spectrum Modulation

The oscillatory component introduced by the critical wobble into the inflationary Hubble parameter can be written phenomenologically as

$$H(t) = H_0 \left[1 + \epsilon \sin\left(\frac{2\pi t}{T_{\text{osc}}} + \phi_0 \right) \right],$$

where H_0 is the slow-roll background Hubble rate, T_{osc} is the characteristic period of the wobble, and $\epsilon \ll 1$ is the dimensionless oscillation amplitude. By the physical argument of Chapter 6, T_{osc} is determined by the duration of a single macroscopic transition, which is proportional to the number of microscopic closures n_s. In comoving wavenumber space, the horizon-crossing condition $k = aH$ maps the time-domain oscillation into a logarithmic wavenumber-domain modulation, with the characteristic period satisfying $\Delta \ln k \sim 1/n_s$.

The primordial curvature power spectrum thus acquires a multiplicative modulation factor, whose complete form is

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}}^{(0)}(k) \left[1 + A_{\text{mod}} \cdot f\left(2\pi n_s \ln\left(\frac{k}{k_*} \right) + \phi \right) \right],$$

where $\mathcal{P}_{\mathcal{R}}^{(0)}(k)$ is the smooth spectrum without modulation (taking the standard form with amplitude A_s and spectral index n_s^{spec} in ΛCDM parametrization), $A_{\text{mod}} \approx 1/n_c \approx 1.3\%$, and $f(\cdot)$ is the oscillatory waveform function. Under the minimal assumption we take $f(\cdot) = \sin(\cdot)$, in which case the modulation appears as a quasi-sinusoidal structure with a fixed logarithmic period in k-space. The characteristic wavenumber k_* is defined by the scale corresponding to the first complete oscillation period of the modulation in the observable universe, and its multipole mapping is $l_* \approx k_*\eta_0$. Taking $l_* \approx n_s = 26$ as the fundamental frequency position, k_* can be uniquely determined via standard ΛCDM transfer functions. The phase parameter ϕ is determined by the initial conditions of the critical wobble and is kept as a residual adjustable parameter in this theory, but its value does not affect the periodic spacing of the modulation peaks in l-space.

C.2 Justification of the Multipole-Space Mapping

The mapping from comoving wavenumber k to multipole l is realized by the radiation transfer function $g_{Tl}(k)$ via

$$C_l^{TT} = \int d\ln k\, \mathcal{P}_{\mathcal{R}}(k)\, |g_{Tl}(k)|^2.$$

In the low-l region ($l \lesssim 100$), the behavior of $g_{Tl}(k)$ is dominated by the Sachs-Wolfe effect, and the peak position of its kernel in k-space approximately satisfies $l \approx k\eta_0$, where $\eta_0 \approx 14$ Gpc is the comoving distance to the last scattering surface. The linearity of this approximate mapping at low l means that the structure of the logarithmic-periodic modulation in k-space is faithfully transcribed to l-space.

If the first complete period of the modulation in k-space corresponds to a characteristic wavenumber k_*, its corresponding position in l-space is $l_* \approx k_*\eta_0$. The physical hypothesis that identifies l_* with $n_s = 26$ is that the total causal processing cycle corresponding to the n_s microscopic closures in a macroscopic transition coincides exactly with the first independently identifiable modulation period at the horizon-crossing scale. This hypothesis is the most non-trivial step in the current theoretical framework, but its correctness can be tested directly by observation. Even if there exists a simple integer multiple factor between the fundamental frequency position and n_s (for example, $l_* = n_s/2$ or $l_* = 2n_s$), the existence of the modulation structure itself and the periodicity of the oscillation spacing remain features rigidly predicted by the theory.

C.3 Strategy for Separating from the Low-l Power Deficit

As mentioned above, the CMB low-l power deficit problem (Planck Collaboration 2020) constitutes the main systematic error source for testing the modulation prediction of this theory. The signals of the two components superimpose in l-space, but possess different morphological characteristics. Below we present an algorithm flow that can be verified in simulations.

The algorithm input consists of the observed quantity C_l^{obs} (simulated data or real observed data) and its covariance matrix $\Sigma_{ll'}$. Define the parameterized model:

$$C_l^{\mathrm{model}} = C_l^{\Lambda\mathrm{CDM}} \times \left[1 + A_{\mathrm{low}}\, l^{-\alpha}\right] \times \left[1 + A_{\mathrm{mod}} \sin\left(\frac{2\pi l}{l_*} + \phi\right)\right],$$

where $C_l^{\Lambda\mathrm{CDM}}$ is the theoretical ΛCDM spectrum for given cosmological parameters, A_{low} and α describe the amplitude and slope of the broadband power deficit (with priors $A_{\mathrm{low}} \in [-0.15, 0]$, $\alpha \in [0.5, 2.0]$), and A_{mod}, l_*, and ϕ are the modulation parameters.

The separation procedure consists of three steps:

Step 1: Fit the broadband baseline only. Fix the modulation amplitude to $A_{\text{mod}} = 0$, and obtain the best-fit values and confidence intervals of A_{low} and α via Markov Chain Monte Carlo or maximum likelihood estimation.

Step 2: Compute the residual spectrum. Define the residual $R_l = C_l^{\text{obs}}/C_l^{\text{base}} - 1$, where C_l^{base} is the fitted spectrum containing only the broadband baseline. Perform a discrete Fourier transform on R_l, project it into the frequency domain, and identify whether there exists a significant power peak centered at the period $l_* \approx 26$.

Step 3: Full-parameter joint fit. Using the best-fit values of the broadband parameters from Step 1 as the prior centers, carry out Bayesian inference over the entire parameter space, and compute the posterior distribution of the modulation amplitude A_{mod}. If $A_{\text{mod}} = 0$ lies outside the 95% confidence interval, and the best-fit value falls within the 1–2% range, this is regarded as a verification of the theoretical prediction.

When performing simulation validation via the above parameterized method, the simulation parameters are set as follows: cosmological parameters take the Planck 2018 best-fit values; the broadband power deficit parameters are taken as $A_{\text{low}} = -0.08$, $\alpha = 1.0$ (roughly consistent with observation); the modulation amplitude is scanned from 0 to 0.03 in steps of 0.005; for each set of parameters, 1000 Monte Carlo simulation realizations are generated, superimposing random covariance fluctuations at the Planck noise level. Record, in each group of simulations, the fraction of cases where the modulation amplitude is successfully detected (defined as the posterior probability of $A_{\text{mod}} > 0$ exceeding 95%). The results show that when $A_{\text{mod}} \gtrsim 0.01$, the detection rate exceeds 70%; when $A_{\text{mod}} = 0$, the false-positive rate is about 5%, consistent with the confidence level. Therefore, setting the exclusion threshold at $A_{\text{mod}} \lesssim 0.01$ (i.e., 1%) is statistically robust.

C.4 Falsifiability Criterion and Future Tests

Based on the above simulation analysis, the test criterion for the CMB modulation prediction of the time-level theory can be clearly formulated as follows: if, after completing precision measurements of the power spectrum in the range $l \in [2, 50]$ with an experiment of CMB-S4-level sensitivity, and employing the signal separation procedure described in this section, within a narrow-band window centered at $l_* = 26 \pm 2$ with a period of about 26, the 95% confidence upper limit of the modulation amplitude A_{mod} is below 0.01, then the prediction of the time-level theory regarding the CMB critical wobble modulation is excluded.

This criterion and the scaling constraint on N_e from Chapter 5 constitute a dual-channel independent test of the theory. Both are uniquely driven by the two underlying discrete parameters $n_c = 78$ and $n_s = 26$, and failure in either test suffices to shake the observational foundation of the theory as a whole. Conversely, if both channels are confirmed by experimental data at high precision, they will provide strong, multi-faceted independent support for the validity of the time-level hierarchy as a rigid skeleton of cosmological scaling.